Theorem: The following is an example of a family of intervals over the real number line $A_n = \{x \in \mathbb{R} : 0 < x < c \in \mathbb{R}\}$ where $c = k - n\epsilon$ for some natural k and real $\epsilon > 0$. If such a family of intervals exist then $A_{n+1} \subset A_n$ for all n and $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

Proof: We will assume a family of intervals A_n exists and prove the properties stated in the theorem. The strategy will be to prove the subset part and then using the Archimedean property of the reals we will prove that the repeated intersection of the families is empty.

To prove a set A_{n+1} is a subset of A_n for all n, we need to show that if $x \in A_{n+1}$ then $x \in A_n$ for all n.

Pick an arbitrary n. Consider A_{n+1}. If $x \in A_{n+1}$ then by definition of A_n, $0 < x < k - (n+1)\epsilon$:(*).

Recall that $\epsilon > 0$. So $-\epsilon < 0$.
Add $k - n\epsilon$ to both sides: $k - n\epsilon - \epsilon = k - (n+1)\epsilon < k - n\epsilon$.
By (*): $0 < x < k - (n+1)\epsilon < k - n\epsilon$. So $x \in A_n$. Since $x \in A_{n+1}$ implies $x \in A_n$ we can conclude that $A_{n+1} \subset A_n$. Since n was chose arbitrarily this logic holds for all n and the first property has been shown.

Reminder of the definition: $A_n = \{x \in \mathbb{R} : 0 < x < c \in \mathbb{R}\}$ where $c = k - n\epsilon$ for some integer k and $\epsilon > 0$. By the Archimedean property, there exists an n such that $k < n\epsilon$. Label this n as m. Its corresponding set is A_m. This means that $k - m\epsilon < 0$. There are no values of x such that $x > 0$ and $x < k - m\epsilon < 0$. A_m has no elements, hence $A_m = \emptyset$.

The definition of $\bigcap_{n=1}^{\infty} A_n$: $\{x|(\forall n)(x \in A_n\}$. Since we found an A_n which does not contain any x (A_m), there are no values of x that are in every A_n so $\bigcap_{n=1}^{\infty} A_n = \emptyset$. This proves the second property and the theorem is proved.